

THE A -THEORETIC FARRELL–JONES CONJECTURE FOR VIRTUALLY SOLVABLE GROUPS

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ABSTRACT. We prove the A -theoretic Farrell–Jones Conjecture for virtually solvable groups. As a corollary, we obtain that the conjecture holds for S -arithmetic groups and lattices in almost connected Lie groups.

1. INTRODUCTION

For every group G there is a functor $\mathbb{A}: \text{Or}G \rightarrow \text{Spectra}$ from the orbit category of G to the category of spectra sending G/H to (a spectrum weakly equivalent to) the non-connective A -theory spectrum $\mathbb{A}(BH)$. For any such functor $\mathbb{F}: \text{Or}G \rightarrow \text{Spectra}$, a G -homology theory $H_{\mathbb{F}}$ can be constructed via

$$H_{\mathbb{F}}(X) := \text{Map}_G(_, X_+) \wedge_{\text{Or}G} \mathbb{F},$$

see Davis and Lück [DL98]. We will denote its homotopy groups by $H_n^G(X; \mathbb{F}) := \pi_n H_{\mathbb{F}}(X)$. The assembly map for the family of virtually cyclic subgroups (in A -theory) is the map

$$H_n^G(E_{\mathcal{VCyc}}G; \mathbb{A}) \rightarrow H_n^G(\text{pt}; \mathbb{A}) \cong A_n(BG)$$

induced by the map $E_{\mathcal{VCyc}}G \rightarrow \text{pt}$. Here, $E_{\mathcal{VCyc}}G$ denotes the classifying space for the family of virtually cyclic subgroups, see Lück [Lüc05]. The assembly map can more generally be defined with coefficients, cf. [UW, Conjecture 7.1]. In this note, we consider the *A -theoretic Farrell–Jones Conjecture with coefficients and finite wreath products*, which predicts for a discrete group G that the assembly map with coefficients is an isomorphism for every wreath product $G \wr F$ of G with a finite group F .

Our main result is the following:

Theorem 1.1. *Let G be a virtually solvable group. Then G satisfies the Farrell–Jones Conjecture for A -theory with coefficients and finite wreath products.*

Using this, we can adapt previous work by R  ping [R  p16] and Kammeyer, L  ck and R  ping [KLR16] to A -theory:

Corollary 1.2. *The A -theoretic Farrell–Jones Conjecture with coefficients and finite wreath products holds for subgroups of $\text{GL}_n(\mathbb{Q})$ or $\text{GL}_n(F(t))$, where F is a finite field.*

In particular, the conjecture holds for S -arithmetic groups.

Proof. The proof works as the one of [R  p16, Theorem 8.13]: Since the conjecture is inherited under directed colimits [ELP⁺, Theorem 1.1(ii)], it suffices to consider linear groups over localizations at finitely many primes. Then [R  p16, Proposition 2.2] together with [ELP⁺, Corollary 6.6] shows that such a group satisfies the conjecture relative to a certain family of subgroups, all whose members in turn

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satisfy the conjecture relative to the class of virtually solvable groups [Rüp16, Theorem 8.12]. The corollary follows from Theorem 1.1 together with the Transitivity Principle [UW, Proposition 11.2]. \square

Corollary 1.3. *The A -theoretic Farrell–Jones Conjecture with coefficients and finite wreath products holds for arbitrary lattices in almost connected Lie groups.*

More generally, it holds for lattices Γ in second countable, locally compact Hausdorff groups G whose group of path components $\pi_0(G)$ is discrete and satisfies the A -theoretic Farrell–Jones Conjecture with coefficients and finite wreath products.

Proof. In [KLR16], it is shown that a class of groups satisfying the list of properties from [KLR16, Theorem 2] also contains the groups considered in the corollary.

The statement of [KLR16, Theorem 2] holds for the class of groups satisfying the A -theoretic Farrell–Jones Conjecture with coefficients and finite wreath products by [ELP⁺, Theorem 1.1], Theorem 1.1 and Corollary 1.2. \square

As explained in [ELP⁺, Section 3], the analogous statements of Theorem 1.1, Corollary 1.2 and Corollary 1.3 for (topological, PL or smooth) Whitehead spectra and pseudoisotopy spectra also hold true.

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2. DRESS–FARRELL–HSIANG–JONES GROUPS

The proof of the A -theoretic Farrell–Jones Conjecture for solvable groups relies on a concoction of the Farrell–Hsiang method [UW] and transfer reducibility [ELP⁺] which mimics the combination of the methods from [BL12b] and [BL12a, Weg12] in [Weg15].

Definition 2.1. Let F be a finite group. We call F a *Dress group* if there exists a normal series $P \trianglelefteq H \trianglelefteq F$ such that P is a p -group for some prime p , H/P is cyclic and F/H is a q -group for some prime q .

We refer to [Weg15, Definition 2.7] and [Weg15, Definition 2.12] for the definitions of “homotopy coherent G -action” and “controlled domination”.

Definition 2.2. Let G be a discrete group and let $S \subseteq G$ be a finite and symmetric generating set of G which contains the trivial element. Let \mathcal{F} be a family of subgroups of G .

Then G is a *Dress–Farrell–Hsiang–Jones group with respect to \mathcal{F}* , or *DFHJ-group (with respect to \mathcal{F})* for short, if there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there is a homomorphism $\pi: G \rightarrow F$ to a finite group with the property that for every Dress subgroup $D \leq F$ there exist

- (1) a compact, contractible metric space X_D such that for every $\varepsilon > 0$ there is an ε -controlled domination of X_D by an at most N -dimensional finite simplicial complex;
- (2) a homotopy coherent G -action Γ_D on X_D ;
- (3) a $\pi^{-1}(D)$ -simplicial complex Σ_D of dimension at most N whose isotropy is contained in \mathcal{F} ;
- (4) a $\pi^{-1}(D)$ -equivariant map $f_D: G \times X_D \rightarrow \Sigma_D$ such that
 - for all $g \in G$, $x \in X_D$ and $s \in S^n$

$$d^1(f_D(g, x), f_D(gs^{-1}, \Gamma_D(s, x))) \leq \frac{1}{n},$$

- for all $g \in G$, $x \in X_D$ and $s_0, \dots, s_n \in S^n$

$$\text{diam}\{f_D(g, \Gamma_D(s_n, t_n, \dots, s_0, x)) \mid t_1, \dots, t_n \in [0, 1]\} \leq \frac{2}{n}.$$

Remark 2.3.

- (1) If G is homotopy transfer reducible with respect to \mathcal{F} [ELP⁺, Definition 6.2], then it is a DFHJ-group with respect to \mathcal{F} : Choose the finite quotient to be trivial for all n .
- (2) If G is a Dress–Farrell–Hsiang group with respect to \mathcal{F} [UW, Definition 7.3], then it is a DFHJ-group with respect to \mathcal{F} : Choose the transfer space X_D to be a point for all n and D .

Remark 2.4. Condition (4) in Definition 2.2 looks a bit different than [Weg15, Definition 4.1]. The difference lies mostly in notation. As we argue in the proof of Proposition 3.3 below, the condition in [Weg15, Definition 4.1] implies ours. Conversely, the proof showing the existence of the functor F in diagram (2.1) (cf. [ELP⁺, Lemma 6.11]) shows that condition (4) also yields the condition in [Weg15, Definition 4.1], up to some constants.

Theorem 2.5. *Suppose that G is a DFHJ-group with respect to a family \mathcal{F} of subgroups of G .*

Then the A -theoretic isomorphism conjecture with coefficients relative \mathcal{F} holds for G .

The remainder of this section is dedicated to a proof of Theorem 2.5 and is modelled on [Weg15, Section 4.2]. Just like the proofs in [UW, ELP⁺], we show that the fiber of the assembly map is weakly contractible. This uses the fact that this fiber can be modelled by the K -theory of certain categories of controlled retractive spaces, whose definition we recall next (cf. also [UW, Sections 2 and 3]).

A *coarse structure* is a triple $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ such that Z is a Hausdorff G -space, \mathfrak{C} is a collection of reflexive, symmetric and G -invariant relations on Z which is closed under taking finite unions and compositions, and \mathfrak{S} is a collection of G -invariant subsets of Z which is closed under taking finite unions. See [UW, Definition 3.23] for the notion of a *morphism of coarse structures*.

Fix a coarse structure \mathfrak{Z} .

A *labeled G -CW-complex relative W* , see [UW, Definition 2.3], is a pair (Y, κ) , where Y is a free G -CW-complex relative W together with a G -equivariant function $\kappa: \diamond Y \rightarrow Z$. Here, $\diamond Y$ denotes the (discrete) set of relative cells of Y .

A \mathfrak{Z} -*controlled map* $f: (Y_1, \kappa_1) \rightarrow (Y_2, \kappa_2)$ is a G -equivariant, cellular map $f: Y_1 \rightarrow Y_2$ relative W such that for all $k \in \mathbb{N}$ there is some $C \in \mathfrak{C}$ for which

$$(\kappa_2, \kappa_1)(\{(e_2, e_1) \mid e_1 \in \diamond_k Y_1, e_2 \in \diamond Y_2, \langle f(e_1) \rangle \cap e_2 \neq \emptyset\}) \subseteq C$$

holds.

A \mathfrak{Z} -*controlled G -CW-complex relative W* is a labeled G -CW-complex (Y, κ) relative W , such that the identity is a \mathfrak{Z} -controlled map and for all $k \in \mathbb{N}$ there is some $S \in \mathfrak{S}$ such that

$$\kappa(\diamond_k Y) \subseteq S.$$

A \mathfrak{Z} -*controlled retractive space relative W* is a \mathfrak{Z} -controlled G -CW-complex (Y, κ) relative W together with a G -equivariant retraction $r: Y \rightarrow W$, ie. a left inverse to the structural inclusion $W \hookrightarrow Y$. The \mathfrak{Z} -controlled retractive spaces relative W form a category $\mathcal{R}^G(W, \mathfrak{Z})$ in which *morphisms* are \mathfrak{Z} -controlled maps which additionally respect the chosen retractions.

The category of controlled G -CW-complexes (relative W) and controlled maps admits a notion of *controlled homotopies*, see [UW, Definition 2.5] via the objects $(Y \times [0, 1], \kappa \circ pr_Y)$, where $Y \times [0, 1]$ denotes the reduced product which identifies

$W \times [0, 1] \subseteq Y \times [0, 1]$ to a single copy of W and $pr_Y : \diamond Y \times [0, 1] \rightarrow \diamond Y$ is the canonical projection. In particular, we obtain a notion of *controlled homotopy equivalence* (or *h-equivalence*).

A \mathfrak{Z} -controlled retractive space (Y, κ) is called *finite* if it is finite-dimensional, the image of $Y \setminus W$ under the retraction meets the orbits of only finitely many path components of W and for each $z \in Z$ there is some open neighborhood U of z such that $\kappa^{-1}(U)$ is finite, see [UW, Definition 3.3].

A \mathfrak{Z} -controlled retractive space (Y, κ) is called *finitely dominated*, if there are a finite \mathfrak{Z} -controlled, retractive space D , a morphism $p : D \rightarrow Y$ and a \mathfrak{Z} -controlled map $i : Y \rightarrow D$ such that $p \circ i$ is controlled homotopic to id_Y .

The finite, respectively finitely dominated, \mathfrak{Z} -controlled retractive spaces form full subcategories $\mathcal{R}_f^G(W, \mathfrak{Z}) \subset \mathcal{R}_{fd}^G(W, \mathfrak{Z}) \subset \mathcal{R}^G(W, \mathfrak{Z})$. All three of these categories support a Waldhausen category structure in which inclusions of G -invariant subcomplexes up to isomorphism are the cofibrations and controlled homotopy equivalences are the weak equivalences, see [UW, Corollary 3.22].

Let X be a G -CW-complex and let M be a metric space with free, isometric G -action. Define $\mathfrak{C}_{bdd}(M)$ to be the collection of all subsets $C \subset M \times M$ which are of the form

$$C = \{(m, m') \in M \times M \mid d(m, m') \leq \alpha\}$$

for some $\alpha \geq 0$. Define further $\mathfrak{C}_{Gcc}(X)$ to be the collection of all $C \subset (X \times [1, \infty]) \times (X \times [1, \infty])$ which satisfy the following:

- (1) For every $x \in X$ and every G_x -invariant open neighborhood U of (x, ∞) in $X \times [1, \infty]$, there exists a G_x -invariant open neighborhood $V \subset U$ of (x, ∞) such that $((X \times [1, \infty]) \setminus U) \times V \cap C = \emptyset$.
- (2) Let $p_{[1, \infty[} : X \times [1, \infty[\rightarrow [1, \infty[$ be the projection map. Equip $[1, \infty[$ with the Euclidean metric. Then there exists some $B \in \mathfrak{C}_{bdd}([1, \infty])$ such that $C \subset p_{[1, \infty[}^{-1}(B)$.
- (3) C is symmetric, G -invariant and contains the diagonal.

Next define $\mathfrak{C}(M, X)$: Let $p_M : M \times X \times [1, \infty[\rightarrow M$ and $p_{X \times [1, \infty[} : M \times X \times [1, \infty[\rightarrow X \times [1, \infty[$ denote the projection maps. Then $\mathfrak{C}(M, X)$ is the collection of all subsets $C \subset (M \times X \times [1, \infty])^2$ which are of the form

$$C = p_M^{-1}(B) \cap p_{X \times [1, \infty[}^{-1}(C')$$

for some $B \in \mathfrak{C}_{bdd}(M)$ and $C' \in \mathfrak{C}_{Gcc}(X)$.

Finally, define $\mathfrak{S}(M, X)$ to be the collection of all subsets $S \subset M \times X \times [1, \infty[$ which are of the form $S = K \times [1, \infty[$ for some G -compact subset $K \subset M \times X$.

All these data combine to a coarse structure

$$\mathbb{J}(M, X) := (M \times X \times [1, \infty[, \mathfrak{C}(M, X), \mathfrak{S}(M, X))$$

which serves to define the “obstruction category” $\mathcal{R}_f^G(W, \mathbb{J}(G, E_{\mathcal{F}}(G))), h)$, cf. [UW, Example 2.2 and Definition 6.1]. The spectrum $\mathbb{F}(G, W, E_{\mathcal{F}}(G))$ alluded to above is the non-connective K -theory spectrum of $\mathcal{R}_f^G(W, \mathbb{J}(G, E_{\mathcal{F}}(G)))$ with respect to the h -equivalences, cf. [UW, Section 5]. By [UW, Corollary 6.11], a group G satisfies the Farrell–Jones Conjecture with coefficients in A -theory with respect to \mathcal{F} if and only if $\mathbb{F}(G, W, E_{\mathcal{F}}(G))$ is weakly contractible for every free G -CW-complex W .

Suppose now that G is a DFHJ-group. By definition, there exists some $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there is a homomorphism $\pi_n : G \rightarrow F_n$ to a finite group with the property that for every Dress subgroup $D \leq F_n$ there exist

- (1) a compact, contractible metric space $X_{n,D}$ such that for every $\varepsilon > 0$ there is an ε -controlled domination of $X_{n,D}$ by an at most N -dimensional finite simplicial complex;
- (2) a homotopy coherent G -action $\Gamma_{n,D}$ on $X_{n,D}$;

- (3) a $\pi_n^{-1}(D)$ -simplicial complex $\Sigma_{n,D}$ of dimension at most N whose isotropy is contained in \mathcal{F} ;
- (4) a $\pi_n^{-1}(D)$ -equivariant map $f_{n,D}: G \times X_{n,D} \rightarrow \Sigma_{n,D}$ such that
- for all $g \in G$, $x \in X_D$ and $s \in S^n$

$$d^{h^1}(f_{n,D}(g, x), f_{n,D}(gs^{-1}, \Gamma_D(s, x))) \leq \frac{1}{n},$$

- for all $g \in G$, $x \in X_D$ and $s_0, \dots, s_n \in S^n$

$$\text{diam}\{f_{n,D}(g, \Gamma_{n,D}(s_n, t_n, \dots, s_0, x)) \mid t_1, \dots, t_n \in [0, 1]\} \leq \frac{2}{n}.$$

Assume we have chosen all of this. Then the proof is organized around the following diagram, in which we abbreviate $E := E_{\mathcal{F}}(G)$ (further explanations follow below):

(2.1)

$$\begin{array}{ccc}
 (\mathcal{R}_f^G(W, \mathbb{J}(G, E)), h) & \xrightarrow{i} & (\mathcal{R}_{fd}^G(W, \mathbb{J}(G, E)), h) \\
 \Delta_f \downarrow & & \Delta_{fd} \downarrow \\
 (\mathcal{R}_f^G(W, \mathbb{J}((G)_n, E)), h) & \xrightarrow{j} & (\mathcal{R}_{fd}^G(W, \mathbb{J}((G)_n, E)), h^{fin}) \\
 \uparrow P_S & \nearrow P_X & \uparrow P_\Sigma \\
 (\mathcal{R}_f^G(W, \mathbb{J}((S_n \times G)_n, E)), h) & & \\
 \text{trans}_2 \downarrow & \nearrow F & \\
 (\mathcal{R}_{fd}^G(W, \mathbb{J}((X_n \times G)_n, E)), h^{fin}) & \xrightarrow{F} & (\mathcal{R}_{fd}^G(W, \mathbb{J}((\Sigma_n \times G)_n, E)), h^{fin})
 \end{array}$$

Diagram (2.1) involves some additional notation which we explain first.

Suppose that $(M_n)_n$ is a sequence of metric spaces with a free, isometric G -action. Let X be a G -CW-complex. Following [UW, Section 7], define the coarse structure

$$\mathbb{J}((M_n)_n, X) := \left(\coprod_n M_n \times X \times [1, \infty[, \mathfrak{C}((M_n)_n, X), \mathfrak{S}((M_n)_n, X) \right)$$

as follows: Members of $\mathfrak{C}((M_n)_n, X)$ are of the form $C = \coprod_n C_n$ with $C_n \in \mathfrak{C}(M_n, X)$, and we additionally require that C satisfies the *uniform metric control condition*: There is some $\alpha > 0$, independent of n , such that for all $((m, x, t), (m', x', t')) \in C$ we have $d(m, m') < \alpha$. Members of $\mathfrak{S}((M_n)_n, X)$ are sets of the form $T = \coprod_n T_n$ with $T_n \in \mathfrak{S}(M_n, X)$. The resulting category $\mathcal{R}^G(W, \mathbb{J}((M_n)_n, X))$ is canonically a subcategory of the product category $\prod_n \mathcal{R}^G(W, \mathbb{J}(M_n, X))$.

Some instances of the category $\mathcal{R}^G(W, \mathbb{J}((M_n)_n, X))$ we consider in diagram (2.1) come equipped with another notion of weak equivalence: Let $(Y_n)_n$ be an object of $\mathcal{R}^G(W, \mathbb{J}((M_n)_n, E))$. For $\nu \in \mathbb{N}$, we denote by $(-)_n > \nu$ the endofunctor which sends $(Y_n)_n$ to the sequence $(\tilde{Y}_n)_n$ with $\tilde{Y}_n = *$ for $n \leq \nu$ and $\tilde{Y}_n = Y_n$ for $n > \nu$. A morphism $(f_n)_n: (Y_n)_n \rightarrow (Y'_n)_n$ is an *h^{fin} -equivalence* if there is some $\nu \in \mathbb{N}$, such that $(f_n)_{n > \nu}: (Y_n)_{n > \nu} \rightarrow (Y'_n)_{n > \nu}$ is an h -equivalence.

Next, we define the families of metric spaces that we plug into the coarse structure $\mathbb{J}(-, E)$. As a shorthand, we denote the preimage $\pi_n^{-1}(D)$ of any Dress group $D \leq F_n$ by \overline{D} .

- (1) The family $(G)_n$ is the constant family in which we equip each component with the word metric on G with respect to S .
- (2) Let $\mathcal{D}r_n$ denote the family of Dress subgroups of F_n . Then define the G -space $S_n := \coprod_{D \in \mathcal{D}r_n} G/\overline{D}$. We equip $S_n \times G$ with the diagonal G -action

and the quasi-metric d_{S_n} given by

$$d_{S_n}((g_1\overline{D}, g_2), (h_1\overline{D'}, h_2)) := \begin{cases} d_G(g_2, h_2) & \overline{D} = \overline{D'}, g_1\overline{D} = h_1\overline{D}, \\ \infty & \text{otherwise.} \end{cases}$$

- (3) The space X_n is defined to be $\coprod_{D \in \mathcal{D}r_n} X_{n,D} \times G/\overline{D}$. Define for each $D \in \mathcal{D}r_n$ the constant $\Lambda_{n,D}$ as in [ELP⁺, Section 6]. We equip $X_n \times G$ with the G -action $\gamma \cdot (x, g_1\overline{D}, g_2) := (x, \gamma g_1\overline{D}, \gamma g_2)$ and the metric d_{X_n} given by

$$d_{X_n}((x, g_1\overline{D}, g_2), (y, h_1\overline{D'}, h_2)) := \begin{cases} d_G(g_2, h_2) + d_{\Gamma_{n,D}, S^n, n, \Lambda_{n,D}}((x, g_2), (y, h_2)) & \overline{D} = \overline{D'}, g_1\overline{D} = g_2\overline{D} \\ \infty & \text{otherwise,} \end{cases}$$

where we use the metric $d_{\Gamma_{n,D}, S^n, n, \Lambda_{n,D}}$ defined in [Weg15, Definition 2.9].

- (4) Finally, Σ_n is defined to be the G -simplicial complex $\coprod_{D \in \mathcal{D}r_n} G \times_{\overline{D}} \Sigma_{n,D}$, equipped with the metric $n \cdot d^{\ell^1}$, where d^{ℓ^1} denotes the ℓ^1 -metric of a simplicial complex.

When crossing one of the above metric spaces with the group G , we regard the resulting space as a metric space by equipping it with the sum of the given metric and the word metric on G . This defines all categories appearing in diagram (2.1).

Let us now define the functors connecting these categories. The functors i and j are the exact inclusions functors from finite to finitely dominated objects. The functors Δ_f and Δ_{fd} are the diagonal functors sending a given object Y to the constant sequence $(Y)_n$. Note that $j \circ \Delta_f = \Delta_{fd} \circ i$. The functors P_S , P_X and P_Σ are induced the projection maps from $S_n \times G$, $X_n \times G$ and $\Sigma_n \times G$ to G . The functor F is induced by the sequence of maps $(f_n: X_n \times G \rightarrow \Sigma_n \times G)_n$, which we define by

$$f_n(x, g_1\overline{D}, g_2) := (g_1, f_{n,D}(g_1^{-1}g_2, x)).$$

The formula uses secretly the identification $G/\overline{D} \times G \cong G \times_{\overline{D}} G$. Using the contracting properties Definition 2.2 (4), one checks that the functor F is well-defined, the proof being completely analogous to [ELP⁺, Lemma 6.11]. Moreover, $P_X = P_\Sigma \circ F$.

We make the following claims:

Proposition 2.6.

- (1) After applying K -theory, the dashed arrow trans_1 exists such that $K_m(\Delta_f) = K_m(P_S) \circ \text{trans}_1$.
- (2) After applying K -theory, the dashed arrow trans_2 exists such that $K_m(j \circ P_S) = K_m(P_X) \circ \text{trans}_2$.
- (3) The K -theory of $(\mathcal{R}_{fd}^G(W, \mathbb{J}((\Sigma_n \times G)_n, E)), h^{fin})$ is trivial.
- (4) $K_m(\Delta_{fd} \circ i)$ is injective for all m .

Theorem 2.5 follows from Proposition 2.6 by an easy diagram chase.

Proof of Proposition 2.6. Claim (1) is an immediate consequence of [UW, Proposition 9.2]. Claim (3) is established in [UW, Section 10]. Claim (4) is [ELP⁺, Lemma 6.12]. So all that is left to show is claim (2).

The map trans_2 arises as a slight modification of the transfer constructed in [ELP⁺, Section 7], whose notation we will also use in the following discussion.

Let $\mathcal{R}_f^G(W, \mathbb{J}((S_n \times G)_n, E))_{\alpha, d}$ denote the subcategory of $\mathcal{R}_f^G(W, \mathbb{J}((S_n \times G)_n, E))$ containing only those objects $(Y_n, \kappa_n)_n$ such that Y_n has dimension at most d and is α -controlled over $S_n \times G$, together with morphisms $(\varphi_n: (Y_n, \kappa_n) \rightarrow (Y'_n, \kappa'_n))_n$ which are *cellwise 0-controlled* in the following sense: Each φ_n is a regular map (ie. it maps open cells onto open cells), and for each cell $c \in \diamond Y_n$, we have $\kappa'_n(\varphi_n(c)) =$

$\kappa_n(c)$. Note that such morphisms automatically satisfy the uniform metric control condition.

Arguing as in [ELP⁺, Section 7.1], we observe that it suffices to construct compatible transfers on each $\mathcal{R}_f^G(W, \mathbb{J}((S_n \times G)_n, E))_{\alpha, d}$ individually.

Let $(Y_n, \kappa_n)_n$ be an object in $\mathcal{R}_f^G(W, \mathbb{J}((S_n \times G)_n, E))_{\alpha, d}$. By the definition of the metric d_{S_n} , the complex Y_n decomposes G -equivariantly as $Y_n = \coprod_{D \in \mathcal{D}r_n} Y_{n, D}$, with $Y_{n, D}$ living over the metric component $G/\pi_n^{-1}(D) \times G$. Let $\kappa_{n, D}$ denote the restriction of κ_n to the set of cells of $Y_{n, D}$. Then define

$$\text{trans}_n^{\alpha, d}(Y_n) := \coprod_{D \in \mathcal{D}r_n} \text{trans}_{X_{n, D}}^{\alpha, d}(Y_{n, D}),$$

cf. [ELP⁺, Definition 7.9]. The control map $\text{trans}_n^{\alpha, d}(\kappa_n)$ of $\text{trans}_n^{\alpha, d}(Y_n)$ is defined as in *loc. cit.* (formula directly before Lemma 7.10), replacing G by $S_n \times G$. Then the obvious analog of [ELP⁺, Lemma 7.10] holds, so that

$$\text{trans}^{\alpha, d}((Y_n, \kappa_n)_n) := (\text{trans}_n^{\alpha, d}(Y_n), \text{trans}_n^{\alpha, d}(\kappa_n))_n$$

is indeed an object in $\mathcal{R}_{fd}^G(W, \mathbb{J}(X_n \times G)_n, E))$. By the obvious analog of [ELP⁺, Lemma 7.11], $\text{trans}^{\alpha, d}$ defines a functor

$$\text{trans}^{\alpha, d}: \mathcal{R}_f^G(W, \mathbb{J}((S_n \times G)_n, E))_{\alpha, d} \rightarrow \mathcal{R}_{fd}^G(W, \mathbb{J}(X_n \times G)_n, E)).$$

Since we leave the $S_n \times G \times E \times [1, \infty[$ -component of each κ_n unchanged, the rest of [ELP⁺, Section 7] carries over to show the existence of the map trans_2 , and thus claim (2). \square

Remark 2.7. In fact, the discussion we have given so far only establishes the vanishing of $K_m(\mathcal{R}_f^G(W, \mathbb{J}(E)), h)$ for $m > 0$. In order to show vanishing in all degrees, we need to consider appropriate deloopings constructed by introducing another metric coordinate \mathbb{R}^k . Since this coordinate remains unchanged throughout, the previous discussion applies verbatim. Cf. also [UW, Section 9] and the discussion in Section 6 of [ELP⁺].

3. PROOF OF THE MAIN THEOREM

As in [Weg15, Section 3], the first step in proving Theorem 1.1 lies in reducing the general theorem to some special cases. For any non-zero algebraic number w , set $G_w := \mathbb{Z}[w, w^{-1}] \rtimes_w \mathbb{Z}$.

Lemma 3.1. *If G_w satisfies the A-theoretic Farrell–Jones Conjecture with coefficients and finite wreath products for every non-zero algebraic number w , then so does every virtually solvable group.*

Proof. We claim that the arguments in [Weg15, Section 3] carry over to A-theory. Indeed, the argument relies only on the following statements about the Farrell–Jones Conjecture with coefficients and finite wreath products:

- (1) The class of groups satisfying the conjecture has the following closure properties [ELP⁺, Theorem 1.1(ii)]:
 - If a group satisfies the conjecture, so does every subgroup.
 - If two groups satisfy the conjecture, so do their direct and free products.
 - If $\{G_i\}_{i \in I}$ is a directed system of groups satisfying the conjecture, so does the colimit.
 - If $p: G \twoheadrightarrow Q$ is an epimorphism, and Q as well as every preimage $p^{-1}(C)$ of virtually cyclic subgroups of Q satisfy the conjecture, so does G .
- (2) The following groups satisfy the conjecture:

- Semidirect products $A \rtimes \mathbb{Z}$ with A torsion abelian: This case follows from the case of hyperbolic groups [ELP⁺, Theorem 1.1(i)], cf. [FL03, Lemma 4.1].
- The wreath product $\mathbb{Z} \wr \mathbb{Z}$: This is, for example, a directed colimit of CAT(0)-groups, and hence satisfies the conjecture by [ELP⁺, Theorem 1.1(i)]. Alternatively, one can argue as in [FL03, Lemma 4.3].
- Virtually abelian groups [UW, Corollary 11.11], [ELP⁺, Theorem 1.1(i)].

For details, we refer to [Weg15, Section 3]. \square

If w is a root of unity, G_w is a virtually abelian group (cf. [Weg15, Lemma 5.32]) and satisfies the A -theoretic Farrell Jones Conjecture with coefficients and finite wreath products by [UW, Corollary 11.11]. So we may assume that w is not a root of unity in the sequel.

We recall some notation from [Weg15, Section 5]. In what follows, we fix a non-zero algebraic number w which is not a root of unity. Let \mathcal{O} be the ring of integers in $\mathbb{Q}(w)$. Define the ring \mathcal{O}_w to be

$$\mathcal{O}_w := \{x \in \mathbb{Q}(w) \mid v_{\mathfrak{p}}(x) \geq 0 \text{ for all prime ideals } \mathfrak{p} \subset \mathcal{O} \text{ with } v_{\mathfrak{p}}(w) = 0\},$$

so that $\mathcal{O} \subseteq \mathcal{O}_w$ and $w, w^{-1} \in \mathcal{O}_w$.

For $s \in \mathbb{N}$ we define $t_w(s) \geq 0$ to be the number determined by

$$t_w(s)\mathbb{Z} = \{z \in \mathbb{Z} \mid w^z \equiv 1 \pmod{s\mathcal{O}_w}\}.$$

Lemma 3.2. *Let q_1, q_2 be prime numbers satisfying $q_1 \neq q_2$ and $v_{\mathfrak{p}}(w) = 0$ for all prime factors \mathfrak{p} of q_1 or q_2 in \mathcal{O} . Let m_1, m_2 be natural numbers.*

Consider the finite group $F := (\mathcal{O}_w/q_1^{m_1}q_2^{m_2}\mathcal{O}_w) \rtimes \mathbb{Z}/t_w(q_1^{m_1}q_2^{m_2})\mathbb{Z}$.

For every Dress group $D \leq F$, there exists $i \in \{1, 2\}$ such that the image of D under the canonical projection $\eta_i: F \twoheadrightarrow \mathcal{O}_w/q_i^{m_i}\mathcal{O}_w \rtimes \mathbb{Z}/t_w(q_i^{m_i})\mathbb{Z}$ is hyperelementary.

Proof. Let D be a Dress subgroup of F . Then D fits into a normal series $P \trianglelefteq H \trianglelefteq D$ such that P is a p -group, D/H is a p' -group, P is normal in D and $|H/P|$ is coprime to both p and p' [Win15, Lemma 5.1].

The prime p cannot be q_1 and q_2 at the same time; without loss of generality, assume that $p \neq q_1$. Set $t := t_w(q_1^{m_1}q_2^{m_2})$ and $t_1 := t_w(q_1^{m_1})$. Consider the normal subgroup $N := q_1^{m_1}\mathcal{O}_w/q_1^{m_1}q_2^{m_2}\mathcal{O}_w \rtimes t_1\mathbb{Z}/t\mathbb{Z}$ and let η_1 denote the projection map

$$\eta_1: F \twoheadrightarrow F/N \cong \mathcal{O}_w/q_1^{m_1}\mathcal{O}_w \rtimes \mathbb{Z}/t_1\mathbb{Z}.$$

Then $\eta_1(P) \cap \mathcal{O}_w/q_1^{m_1}\mathcal{O}_w = \{0\}$ since the latter is a q_1 -group and $p \neq q_1$. Hence, $\eta_1(P)$ is mapped isomorphically to a subgroup of $\mathbb{Z}/t_1\mathbb{Z}$ by the projection map $\mathcal{O}_w/q_1^{m_1}\mathcal{O}_w \rtimes \mathbb{Z}/t_1\mathbb{Z} \twoheadrightarrow \mathbb{Z}/t_1\mathbb{Z}$. So $\eta_1(P)$ is cyclic. Since p is coprime to $|H/P|$ and H/P is cyclic, the image $\eta_1(H)$ is also cyclic. It follows that $\eta_1(D)$ is hyperelementary. \square

Proposition 3.3. *Let $w \neq 0$ be an algebraic number which is no root of unity. Then $G_w = \mathbb{Z}[w, w^{-1}] \rtimes \mathbb{Z}$ is a DFHJ-group with respect to the family of virtually abelian subgroups.*

Proof. Let N be the natural number determined by [Weg15, Proposition 5.26]. Let $S \subseteq G_w$ be a finite, symmetric generating set containing the trivial element.

In the proof of [Weg15, Proposition 5.33], it is shown that for every $n \in \mathbb{N}$ and for every sufficiently large prime number q (depending on n) there is a natural number $m \in N$ such that for every hyperelementary subgroup

$$H \leq F_n := \mathcal{O}_w/q^m\mathcal{O}_w \rtimes \mathbb{Z}/t_w(q^m)\mathbb{Z}$$

there exist

- (1) a compact, contractible metric space $X_{n,H}$ such that for every $\varepsilon > 0$ there is an ε -controlled domination of $X_{n,H}$ by an at most N -dimensional finite simplicial complex;¹
- (2) a homotopy coherent G_w -action $\Psi_{n,H}$ on $X_{n,H}$;
- (3) a positive real number $\Lambda_{n,H}$;
- (4) a $\alpha_n^{-1}(H)$ -simplicial complex $E_{n,H}$ of dimension at most N whose isotropy groups are virtually cyclic or abelian;
- (5) a $\alpha_n^{-1}(H)$ -equivariant map $f_{n,H}: G_w \times X_{n,H} \rightarrow E_{n,H}$ such that

$$n \cdot d^1(f_{n,H}(g, x), f_{n,H}(h, y)) \leq d_{\Psi_{n,H}, S^n, n, \Lambda_{n,H}}((g, x), (h, y))$$

for all $(g, x), (h, y) \in G_w \times X_{n,H}$ with $h^{-1}g \in S^n$.

Here, $\alpha_n: G_w \rightarrow F_n$ denotes the composition of the inclusion $G_w \hookrightarrow \mathcal{O}_w \rtimes \mathbb{Z}$ with the quotient map $\mathcal{O}_w \rtimes \mathbb{Z} \twoheadrightarrow F_n$. The metric $d_{\Psi_{n,H}, S^n, n, \Lambda_{n,H}}$ on $G_w \times X_{n,H}$ is defined in [Weg15, Definition 2.9]. It has the property

$$d_{\Psi_{n,H}, S^n, n, \Lambda_{n,H}}((g, x), (g(s_n \cdots s_0)^{-1}, \Psi_{n,H}(s_n, t_n, \dots, s_0, x))) \leq 1$$

for all $g \in G_w$, $x \in X_{n,H}$ and $s_0, \dots, s_n \in S^n$. Hence,

$$d^1(f_{n,H}(g, x), f_{n,H}(gs^{-1}, \Psi_{n,H}(s, x))) \leq \frac{1}{n}$$

for all $g \in G_w$, $x \in X_{n,H}$ and $s \in S^n$, and

$$\text{diam}\{f_{n,H}(g, \Psi_{n,H}(s_n, t_n, \dots, s_0, x)) \mid t_1, \dots, t_n \in [0, 1]\} \leq \frac{2}{n}$$

for all $g \in G_w$, $x \in X_{n,H}$ and $s_0, \dots, s_n \in S^n$.

Now let us come to the actual proof. For a given $n \in \mathbb{N}$ we choose two distinct (large) prime numbers q_1, q_2 with appropriate natural numbers $m_1, m_2 \in N$ (as described above). Consider the finite group

$$F := \mathcal{O}_w / q_1^{m_1} q_2^{m_2} \mathcal{O}_w \rtimes \mathbb{Z} / t_w(q_1^{m_1} q_2^{m_2}) \mathbb{Z}.$$

Let $D \leq F$ be a Dress subgroup. By Lemma 3.2, there exists $i \in \{1, 2\}$ such that $\eta_i(D)$ is hyperelementary. We have a finite group $F_n := \mathcal{O}_w / q_i^{m_i} \mathcal{O}_w \rtimes \mathbb{Z} / t_w(q_i^{m_i}) \mathbb{Z} = \text{im}(\eta_i)$ with a hyperelementary subgroup $H := \eta_i(D) \leq F_n$. As mentioned at the beginning of the proof, we obtain a homotopy coherent G_w -action $\Gamma_{n,H}$ on a metric space $X_{n,H}$, an $\alpha_n^{-1}(H)$ -simplicial complex $E_{n,H}$ and an $\alpha_n^{-1}(H)$ -equivariant map $f_{n,H}$ with the properties described above. We define $\pi: G_w \rightarrow F$ as the composition of the inclusion $G_w \hookrightarrow \mathcal{O}_w \rtimes \mathbb{Z}$ with the quotient map $\mathcal{O}_w \rtimes \mathbb{Z} \twoheadrightarrow F$. Then $\pi^{-1}(D)$ is a subgroup of $\alpha_n^{-1}(H)$. We finally set $X_D := X_{n,H}$, $\Gamma_D := \Psi_{n,H}$, $\Sigma_D := E_{n,H}$, $f_D := f_{n,H}$. \square

Since virtually abelian groups satisfy the A -theoretic Farrell-Jones Conjecture with coefficients and finite wreath products, Theorem 1.1 follows from Lemma 3.1, Proposition 3.3 and Theorem 2.5 together with the Transitivity Principle [UW, Proposition 11.2] in view of the following:

Lemma 3.4. *Suppose that G is a DFHJ-group with respect to the family of all subgroups which satisfy the A -theoretic Farrell-Jones Conjecture with coefficients and finite wreath products. Let F be a finite group.*

Then $G \wr F$ is a DFHJ-group with respect to the family of all subgroups which satisfy the A -theoretic Farrell-Jones Conjecture with coefficients and finite wreath products.

¹In the proof of [Weg15, Proposition 5.33] the space $X_{n,H}$ is denoted by X_w^R .

Proof. The proof is analogous to that of [Weg15, Lemma 4.3], replacing “hyperelementary” by “Dress” and using the fact that the collection of Dress groups is also closed under taking subgroups and quotients. \square

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